Effect of Quantum Fluctuations on the Dipolar Motion of Bose-Einstein Condensates in Optical Lattices

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We reexamine dipolar motion of condensate atoms in one-dimensional optical lattices and harmonic magnetic traps including quantum fluctuations within the truncated Wigner approximation. In the strong tunneling limit we reproduce the mean field results with a sharp dynamical transition at the critical displacement. When the tunneling is reduced, on the contrary, strong quantum fluctuations lead to finite damping of condensate oscillations even at infinitesimal displacement. We argue that there is a smooth crossover between the chaotic classical transition at finite displacement and the superfluid-to-insulator phase transition at zero displacement. We further analyze the time dependence of the density fluctuations and of the coherence of the condensate and find several nontrivial dynamical effects, which can be observed in the present experimental conditions.

The study of Bose-Einstein condensates of ultracold atoms has been growing rapidly in recent years [1]. Loading bosonic atoms into an optical lattice and enhancing the laser intensity, it is possible to strongly suppress the kinetic energy of the atoms resulting in a superfluid-to-Mott-insulator (SF-MI) quantum phase transition [2,3]: in the strong tunneling limit the condensate is in the superfluid phase with finite phase stiffness, while it is driven to the Mott-insulator phase and loses the phase coherence because of quantum fluctuations if the tunneling becomes small enough. Typical interference patterns obtained in time-of-flight experiments cannot provide sufficient information to see a sharp transition boundary [3,4].

In one-dimensional systems the condensate dynamics has been observed in a dipolar motion experiment [5], where the center-of-mass (c.m.) of the condensate oscillates after a sudden displacement of the magnetic trap with respect to the optical lattice. This dynamics can be described by the mean field Gross-Pitaveskii equations (GPE) in the strong tunneling regime [5,6]. When the displacement is beyond a certain critical value, the condensate oscillations are overdamped [7] and the motion becomes completely decoherent, indicating a classical localization transition [8,9]. Such a transition was recently observed experimentally [10]. It is plausible that as the quantum fluctuations increase, there is a smooth crossover between the quantum (SF-MI) and the classical transitions. However, we emphasize that the first one is a second order transition characterized by completely reversible phase coherence if the system is driven to the insulating phase and then back to the SF [3]. On the other hand, the classical transition is irreversible [9] due to the chaotic excitations inside the condensate cloud.

Motivated by this interesting and important question, in this Letter we investigate the quantum fluctuation effects on the c.m. motion of a condensate in a parabolic potential in a one-dimensional optical lattice using the truncated Wigner approximation (TWA). In the strong tunneling (or weakly interacting) regime, we reproduce the sharp dynamical transition of a mean field calculation [9] with a critical displacement proportional to the square root of the tunneling amplitude. When the tunneling is weak the quantum fluctuations strongly modify the dynamics of the condensate in the following ways: (i) the c.m. motion is damped even for an infinitesimal initial displacement; (ii) the oscillations become overdamped at a critical displacement, D_c, which is below the classical value at the same tunneling; (iii) the condensate dipolar motion is frictionless from one end to another with the period independent of damping, while the superfluid fraction and the phase coherence drop when the condensate passes through the center of the harmonic well. (iv) We further show that at a given damping, the loss of coherence is less for smaller displacements (or stronger quantum fluctuations). So as we increase quantum fluctuations the localization transition becomes more reversible. Our results hence suggest that there is a smooth crossover between the classical localization transition and the quantum superfluid-to-insulator (SF-IN) transition [11] as the displacement goes to zero.

The truncated Wigner approximation [12] has been well known in quantum optics for a while. Recently, it was applied to the systems of interacting bosons [13,14]. In Ref. [14] it was argued that TWA is equivalent to the semiclassical approximation and naturally appears in the quantum expansion of the time evolution of the system. The idea behind TWA is that the expectation value of an observable Ω can be found according to

$$\langle \Omega(t) \rangle \approx \int d\psi_0 d\psi_0^\star \, P(\psi_0, \psi_0^\star) \Omega_{cl}[\psi(t), \psi^\star(t), t], \quad (1)$$

where ψ and $\psi^\star$ denote bosonic fields obeying classical discrete Gross-Pitaevskii equations of motion [15,16]:

$$i\frac{\partial \psi_j}{\partial t} = -J(\psi_{j-1} + \psi_{j+1}) + \frac{K}{2}j^2\psi_j + \frac{U}{2}|\psi_j|^2\psi_j \quad (2)$$

with the initial conditions $\psi(t_0) = \psi_0$ and $\psi^\star(t_0) = \psi_0^\star$. The parameters J, K, and U in Eq. (2) denote the tunneling constant, the harmonic trap curvature, and the on-site interaction, respectively. The function $P(\psi_0, \psi_0^\star)$ is a Wigner transform of the initial density matrix, and can be interpreted as the probability of having particular initial conditions [17]. Finally Ω_{cl} is the Weyl symbol of the operator Ω evaluated on the classical fields $\psi(t)$ and $\psi^\star(t)$. It is important to realize that TWA considerably improves Bogoliubov's theory, especially if the classical dynamics becomes unstable [14]. The way we implement TWA in this Letter is outlined in Ref. [14].

In a dipolar motion experiment, the condensate is initially ($t < 0$) prepared in the superfluid ground state. At $t = 0$ the trap position is suddenly displaced by the distance D_0 from the origin and the condensate starts to move. To find the Wigner transform $P(\psi_0, \psi_0^*)$ of the interacting ground state, we start from the noninteracting Hamiltonian ($U = 0$), where the function $P(\psi_0, \psi_0^*)$ can be trivially computed, and then adiabatically increase U to the actual value [14]. We check that the increase of the interaction is slow enough so that the final result is not affected by this procedure. The further implementation of Eq. (1) is based on the Monte Carlo averaging of the $\Omega_{cl}[\psi(t), \psi^\star(t), t]$ over the different initial conditions weighted by the probability $P(\psi_0, \psi_0^*)$.

In Fig. 1 we show the calculated dynamical phase diagram in terms of the inverse tunneling $1/J$ and the displacement D_0. The contours correspond to the constant damping of the c.m. oscillations, $\gamma \equiv \ln(D_0/D_1)$, where
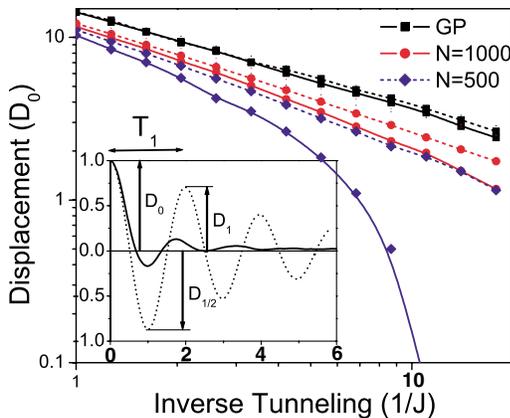

FIG. 1 (color online). Dynamical phase diagram for $UN = 50$, and $K = 0.02$ (the occupancy of the central site, N_0, is approximately 5% of the total number of bosons, N). The solid and the dashed lines correspond to damping $\gamma = 0.11$ and $\gamma = 0.36$, respectively. The large separation between the two curves for stronger quantum fluctuations (smaller N) implies broadening of the transition. The inset shows the typical temporal c.m. oscillations for $\gamma \approx 0.36$ (dashed line) and $\gamma \approx 2.1$ (solid line).

D_0 and D_1 are the c.m. positions at $t = 0$ and $t = T_1$ (see the inset). The solid and the dashed lines correspond to the damping of $\gamma = 0.11$ and 0.36, respectively [18]. The top two curves correspond to the Gross-Pitaevskii result ($D_c \propto \sqrt{J}$), where the transition is so sharp that they almost coincide. When we increase quantum fluctuations reducing the total number of bosons N (but keeping the product UN the same, which is equivalent to fixing the chemical potential [19]), we find: (i) for a given J, the damping of the oscillations occurs at a smaller displacement than in the classical case, and the transition becomes broader for smaller N. (ii) If the tunneling is small enough (say $J^{-1} > 10$ for $N = 500$), then even an infinitesimal displacement results in considerable damping of the c.m. motion. Although because of the time-consuming computation we are able to trace only $\gamma = 0.11$ contour for $N = 500$ to zero, it is clear that such a behavior should be generally true for any given γ, if J becomes small enough. For the parameters chosen in Fig. 1, the SF-IN transition [11] occurs at $J_c^{-1} \sim N_0/U \sim 10^3$. This is much larger than the range of tunneling where we perform the calculations so that the results we obtained in this Letter within TWA are expected to be fairly reliable. We anticipate that contours of larger damping will terminate much closer to the SF-IN transition. In Fig. 2 we show the damping of the c.m. motion as a function of the inverse tunneling at $D_0 = 5$. Because of the finite size effects, GP solution also has some finite damping close to the transition point ($\gamma = \infty$). Clearly the damping becomes broader and the transition point shifts to the lower tunneling when quantum fluctuations are enhanced. To affirm the quantum effects in the thermodynamic limit, we also show the results (dotted lines) for a twice as large system ($K \rightarrow K/4$ and $N \rightarrow 2N$, keeping the same chemical potential [19]). The displacement was also scaled by a factor of 2 in order to fix the same
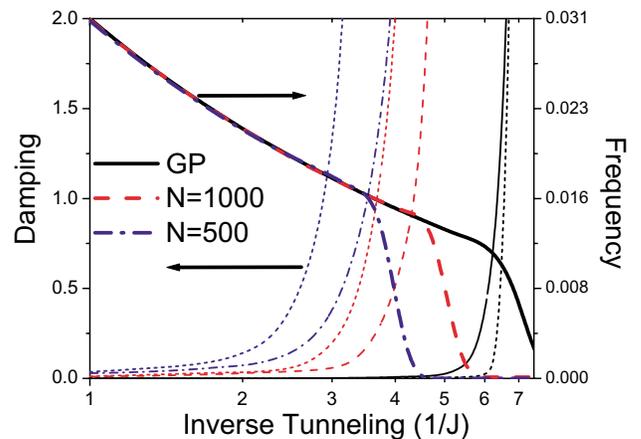

FIG. 2 (color online). Damping (thin lines) and the oscillation frequency (thick lines) vs inverse tunneling for $D_0 = 5$. Dotted lines show damping for a twice as large system with the displacement also scaled by a factor of 2 (see the text).

condensate maximum velocity. We find that the damping of the GP solution becomes a sharper function of the tunneling (but diverges at the same tunneling amplitude) while the quantum c.m. motion is destroyed at even larger tunneling. This result reflects the fact that the GP solution has only one length scale associated with the condensate size. In the quantum case there appears another scale, associated with generation of quasiparticles out of the condensate. Therefore, the finite damping of the c.m. motion shown in Figs. 1 and 2 exists even in the thermodynamic limit if the condensate velocity (proportional to $D\sqrt{K}$) is kept finite. In Fig. 2, we also show the dependence of the oscillation frequency [18] (ω) on tunneling. Surprisingly, we find that ω does not deviate from the GP result even for the relatively strong damping as long as the c.m. motion remains underdamped.

In Fig. 3 we show the standard deviation of the boson density, $\delta n \equiv N^{-1}\sqrt{\sum_j(\langle N_j^2\rangle - \langle N_j\rangle^2)}$, and the phase coherence, $C = N^{-1}\mathrm{Max}_q[\sum_{jl}\langle \psi_j^\dagger \psi_l\rangle, e^{i(j-l)q}]$, as a function of time for $N = 1000$, $D_0 = 5$ and different tunneling amplitudes. We first note that both δn and C only slightly depend on time if the condensate is near the turning points (i.e., at time/period = 0.5, 1, 1.5, ...), while they change significantly when the condensate has its maximum velocity at the center of the parabolic well (time/period = 0.25, 0.75, 1.25, ...). The sharp increase of the density fluctuations (and the sharp loss of the coherence) can be interpreted as fast generation of the incoherent quasiparticles with a simultaneous decrease of the superfluid fraction. Combining this with the fact that the oscillation frequency does not deviate from its classical (GP) value even for the strong damping, we argue that the whole c.m. motion can be interpreted within Landau's two-fluid model [20]: the superfluid component oscillates frictionlessly and is well described by GP equa-
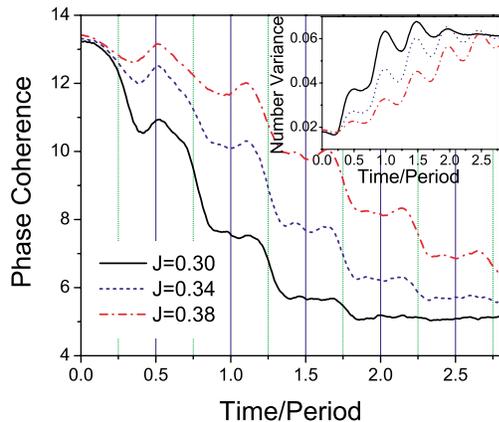
tions, while the normal fluid component has a strong damping and becomes localized near the center of the parabolic well. The former ensures the same oscillation frequency as the classical (GP) value, while the latter causes the damping of the amplitude of the c.m. motion. The generation of quasiparticles continues in each cycle of the oscillations until the coherent motion stops (Fig. 3). If the initial superfluid fraction is too small due to the large quantum depletion, the quasiparticle generation can be so efficient that the c.m. motion becomes overdamped ($\gamma = \infty$).

We can qualitatively understand quantum effects on the dynamical transition using a schematic plot of the probability distribution of the phase difference between two nearest sites as shown in Fig. 4. Given the same tunneling amplitude [Fig. 4(a)], the classical distribution of phase gradients peaks at the c.m. momentum, v. In the mean field picture, the c.m. motion of the condensate is frictionless [9] if v is smaller than the critical value, $v_c = \pi/2$, while it is completely destroyed if $v > v_c$. If we include quantum fluctuations, then the local phase gradient distribution becomes broader, and therefore there is a finite probability to have a large phase difference in a given link even if $v < v_c$, which results in the finite damping of the c.m. motion. Similarly, at a given v [Fig. 4(b)] smaller values of J lead to a broader distribution of the phase gradient, and hence to larger damping.

Now let us look closer to the connection between the classical transition at finite displacement and the quantum SF-IN transition [11] at zero displacement. In Fig. 5, we show the phase coherence as a function of time for different initial displacements but at the same damping, $\gamma = 0.36$. We note that the coherence saturates at a *finite* value, which is closer to the initial one for stronger quantum fluctuations (smaller J and D_0) as shown in the inset. These results suggest that along the $\gamma = \infty$ contour the coherence loss should also approach zero if the displacement gets smaller [21], indicating a smooth crossover between the dynamical irreversible transition at finite displacement and the reversible quantum phase transition at zero displacement. We note that such a dipolar motion can be also used to investigate the superfluid-to

FIG. 3 (color online). Phase coherence (main) and density fluctuations (insert) versus time for the same displacement ($D_0 = 5$) and different tunneling constants. Note that both quantities have been rescaled to be the same at $t = 0$ for the convenience of comparison.
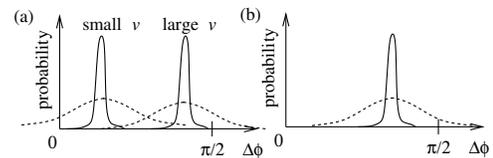

FIG. 4. Schematic figures of the distribution of the phase gradient for the classical (solid) and quantum (dashed) initial states for (a) the same tunneling and different displacements, and for (b) the same displacement and different tunneling. The sharp peak distribution for the classical case indicates that the local velocity of the condensate does not fluctuate; $\pi/2$ is the critical phase gradient for the classical transition.
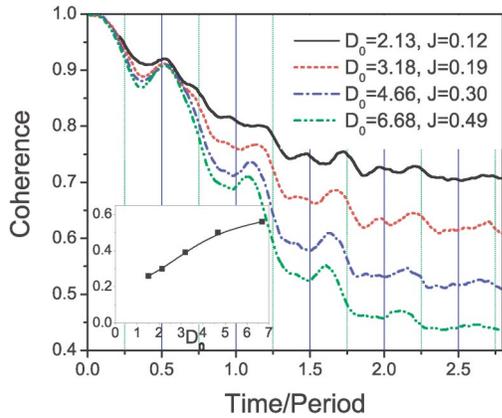
FIG. 5 (color online). Phase coherence (renormalized to the same value at $t = 0$) versus time for different initial displacement D_0 (or the associate tunneling J as indicated) with the same damping $\gamma = 0.36$ of $N = 1000$ (i.e., along the dashed line with circles of Fig. 1). The inset shows the loss of coherence $[(C(t = 0) - C(t = \infty))/C(t = 0)]$ of these data as a function of the displacement. Clearly smaller D_0 (and hence J) results in a smaller loss of coherence of the condensate.

Bose glass insulating phase when a bound random potential is applied [22].

Our analysis also implies that in uniform systems quantum fluctuations lead to the damping of the condensate current state even if the velocity is below the GP critical value [9]. A similar effect was predicted for the current decay in quasi-one-dimensional superconductors due to thermal fluctuations [23], and more recently for the atomic system with a moving defect [24]. We will investigate mechanisms of the current decay in uniform systems in separate publications. The results predicted in this Letter can be directly tested experimentally. For example, it should be possible to observe the damping of the condensate motion at small displacements (Fig. 2) and the ladderlike structure in the coherence or in the number variance (Fig. 3) as a function of time.

In summary, we studied the effects of quantum fluctuations on the dynamical properties of atomic condensates in 1D optical lattices with a parabolic confinement potential. We showed that the quantum fluctuations are very important even *far from* the superfluid-to-insulator transition boundary. We further demonstrate that the dynamical localization transition, which has a purely classical origin, can be smoothly connected with the static quantum SF-MI phase transition. Our results give a number of predictions on the condensate dynamics which can be directly tested in experiments.

[1] See, for example, the special issue in Nature (London), **416**, 206 (2002).
[2] D. Jaksch, *et al.*, Phys. Rev. Lett. **81**, 3108 (1998).
[3] M. Greiner, *et al.*, Nature (London) **415**, 39 (2002).
[4] R. Roth and K. Burnett, Phys. Rev. A **67**, 031602(R) (2003).
[5] F. S. Cataliotti *et al.*, Science **293**, 843 (2001).
[6] M. Kramer, *et al.*, Phys. Rev. Lett. **88**, 180404 (2002).
[7] F. S. Cataliotti, *et al.*, New J. Phys. **5**, 71 (2003).
[8] B. Wu and Q. Niu, Phys. Rev. A **64**, 061603(R) (2001).
[9] A. Smerzi *et al.* Phys. Rev. Lett. **89**, 170402 (2002).
[10] L. Fallani, *et al.*, cond-mat/0404045.
[11] We note that due to the inhomogeneity of the system, it is hard to define a Mott insulator and its transition to the superfluid. Therefore we suggest that the overdamped regime at zero displacement should be associated with the insulator regime in a more general sense.
[12] D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer-Verlag, Berlin, 1994).
[13] M. J. Steel *et al.*, Phys. Rev. A **58**, 4824 (1998); A. Sinatra *et al.*, J. Mod. Opt. **47**, 2629 (2000); Phys. Rev. Lett. **87**, 210404 (2001); J. Phys. B **35**, 3599 (2002).
[14] A. Polkovnikov Phys. Rev. A, **68**, 033609 (2003); **68**, 053604 (2003).
[15] A. Trombettoni and A. Smerzi, Phys. Rev. Lett. **86**, 2353 (2001).
[16] A. Polkovnikov, S. Sachdev, and S. M. Girvin, Phys. Rev. A **66**, 053607 (2002).
[17] This interpretation is not precise, since P is not positively defined. For details, see Ref. [14].
[18] We emphasize that the damping γ and frequency ω defined in this Letter are *not* obtained by fitting the c.m. motion (inset of Fig. 1) by $e^{-\gamma t}\cos(\omega t)$, but by the position ($D_1$) and time ($T_1$) of the first full oscillation cycle [$\gamma \equiv \ln(D_0/D_1)$ and $\omega \equiv 2\pi/T_1$]. This definition helps us to address the frictionless motion of the superfluid component as discussed in the text. However, similar to any other definition, this one becomes ambiguous when γ is large, because the c.m. motion is strongly suppressed even during the first half period (i.e., $D_0 \gg D_{1/2} \sim D_1$).
[19] Note that in this weak tunneling regime, the condensate size is not sensitive to the tunneling (so-called Thomas Fermi limit). The density profile can be estimated by $N_j = N_0(1 - Kj^2/\mu)$ where N_0 is the number of atoms in the central well and the chemical potential $\mu = UN_0$ is determined by the total number of bosons N.
[20] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Oxford University Press, New York, 1987).
[21] We note that the coherence loss during the dynamical process is best defined by the value after adiabatically tuning the interaction U to zero. But the results we present here are clear enough to show its qualitative behavior.
[22] B. Damski *et al.*, Phys. Rev. Lett. **91**, 080403 (2003); D.-W. Wang *et al.*, Phys. Rev. Lett. **92**, 076802 (2004).
[23] J. S. Langer and V. Ambegaokar, Phys. Rev. **164**, 498 (1967); D. E. McCumber and B. I. Halperin, Phys. Rev. B **1**, 1054 (1970).
[24] H. P. Büchler, V. B. Geshkenbein, and G. Blatter, Phys. Rev. Lett. **87**, 100403 (2001).