# Gravity: from Classical, Semi-Classical, Stochastic to Quantum

林 世 昀 Shih-Yuin Lin
National Center for Theoretical Sciences

June 1, 2007 @ NTHU

# Outline

I. Classical Gravity

II. Semiclassical Gravity

III. Stochastic to Quantum Gravity

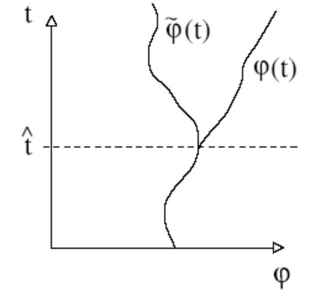
IV. Concluding Remarks

# Prologue

- Theoretical Science: logical explanation, falsifiable prediction

- Theoretical Physics:
  (mathematical, quantitative) descriptions about
  (natural) phenomena
  (repeatable by arranging experiments)

- Physical laws are covariant to coordinate transformation
  (have the same form for each observer)
  Physical quantities are "gauge invaeiant" (Dirac)



# I. Classical Gravity

# I. Classical Gravity

- Newtonian particle mechanics:

  Galilean invariant (equivalence of inertial frame)

$$
\begin{aligned}
\vec{x'} &= \vec{x} - \vec{v}t \\
t' &= t
\end{aligned}
$$

$$
m_i \frac{d\vec{v}_i}{dt} = -\nabla_i \sum_j V_{ij}(|\vec{x}_i - \vec{x}_j|) \quad \rightarrow \quad m_i \frac{d\vec{v'}_i}{dt} = -\nabla'_i \sum_j V_{ij}(|\vec{x'}_i - \vec{x'}_j|)
$$

- Electromagnetic (EM) wave:

  Wave equations are Lorentz invariant

$$
\begin{aligned}
\vec{x'} &= \vec{x} + \frac{(\gamma - 1)}{\beta^2}(\vec{\beta} \cdot \vec{x})\vec{\beta} - \gamma\vec{\beta}x_0 \\
x'_0 &= \gamma(x_0 - \vec{\beta} \cdot \vec{x})
\end{aligned}
$$

$$
x_0 = ct, \quad \vec{\beta} = \frac{\vec{v}}{c}, \quad \beta = |\vec{\beta}|, \quad \gamma = (1 - \beta^2)^{-1/2}
$$

$$
\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi(x) = 0 \quad \rightarrow \quad \left(\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right)\psi(x') = 0
$$

EM wave propagates in constant speed in vacuum.

# I. Classical Gravity

- Particle Electrodynamics

  Interplay between electromagnetic (EM) field

  and sources (charged particles)

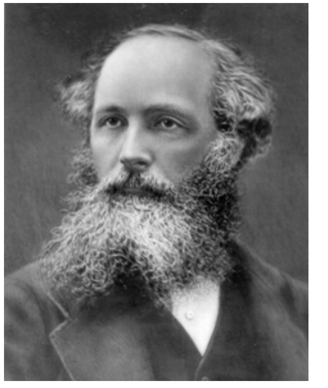
  Maxwell's equations (Maxwell 1873; arr. Heaviside 1884)

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} = \frac{4\pi}{c}\mathbf{j} \qquad \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} = 0$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho \qquad \nabla \cdot \mathbf{B} = 0$$

  Lorentz force density (Lorentz 1892)

$$\mathbf{f} = \rho\mathbf{E} + \frac{1}{c}\mathbf{j} \times \mathbf{B}$$



James Clerk Maxwell (1831~1879) in his 40s.

Hendrik Antoon Lorentz (1853~1928), Nobel prize 1902 (with Zeeman)

# I. Classical Gravity

Newton's gravity law

$$F = G\,\frac{m_1\;m_2}{d^2}$$

action-at-a-distance ~ simultaneity
(violation of cartesian prescriptions)

Coulomb's law (EM force)

$$F = k\,\frac{q_1\;q_2}{d^2}$$

only good in static limit
mediated by EM field (Faraday)

- Einstein's special relativity (1905)
  Introducing Lorentz symmetry to particle mechanics
  while keeping equivalence of inertial frame
  - dropping observer-independent simultaneity
  - postulating c=const, independent of the motion of its source.

  $\Rightarrow$ consistent particle electrodynamics

# I. Classical Gravity

Newton's gravity law

$$F = G\,\frac{m_1\ m_2}{d^2}$$

action-at-a-distance ~ simultaneity
(violation of cartesian prescriptions)

Coulomb's law (EM force)

$$F = k\,\frac{q_1\ q_2}{d^2}$$

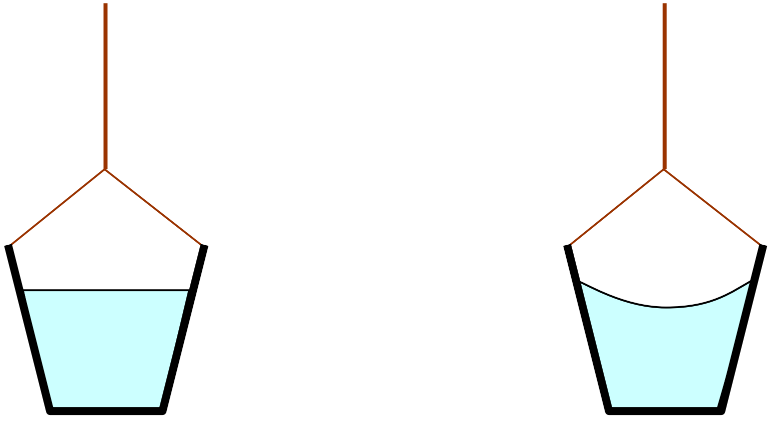
only good in static limit
mediated by EM field (Faraday)

- Einstein's next goal:

  Field theory of gravity with Newton's law as the static limit.

# I. Classical Gravity

- Newton's bucket (*Principia*): against Aristotle, Descartes, Leibniz...



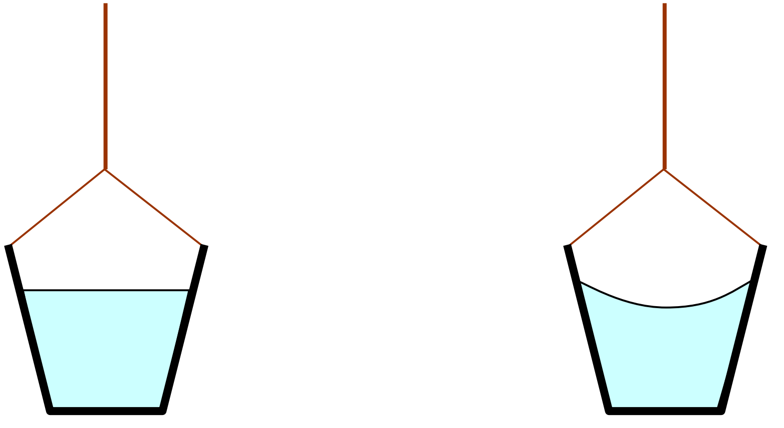
Newton: Water rotates with respect to an absolute space.

absolute motion

However, even Galilean relativity - the velocity of single particle has no physical meaning - does not respect these absolute ideas.

# I. Classical Gravity

- Newton's bucket (*Principia*): against Aristotle, Descartes, Leibniz...

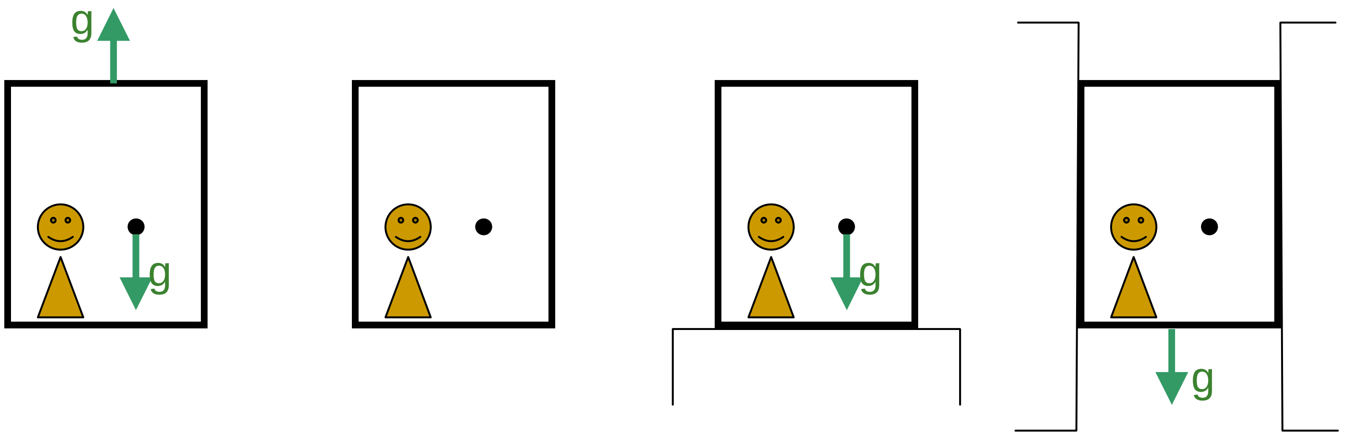


Newton: Water rotates with respect to an absolute spacetime.

Einstein: Water rotates with respect to the gravitational field !!

# I. Classical Gravity

- elevator argument




Newton: inertial mass = active gravitational mass = passive gravitation mass

- Equivalence principle

  A frame linearly accelerated relative to an inertial frame in special relativity is locally identical to a frame at rest in a gravitational field.

# I. Classical Gravity

inertial frame

$$\frac{d^2x^\mu}{d\tau^2} = 0$$

general frame

$$\frac{d^2x^\mu}{d\tau^2} + \underbrace{\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}}_{\text{inertial force}} = 0$$

- $X^I$ : local inertial frame around Event A, $X^I$(A)=0 ($I$ =0,1,2,3)
  $x^\mu$ : general coordinate around A

$$X^I(x) = x^\mu \left.\frac{\partial X^I}{\partial x^\mu}\right|_{x=x(A)} \equiv e^I_\mu(x(A))x^\mu$$

nontrivial "tetrad" ~ gravity

- metric tensor (not applicable to spinor)

$$g_{\mu\nu}(x) = \sum_{I,J=0,1,2,3} \eta_{IJ}e^I_\mu(x)e^J_\nu(x) \equiv \eta_{IJ}e^I_\mu(x)e^J_\nu(x)$$

Gravitational "field"

$$\eta_{IJ} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

# I. Classical Gravity

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

- Manifold endowed with a metric: Riemannian manifold

  Example: In $R^3$ , one may choose

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

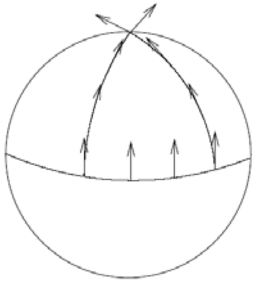
$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

inverse $\quad g_{ij} g^{jk} = \delta_i{}^k \quad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/r^2\sin^2\theta \end{pmatrix}$

covariant derivative of vectors $\quad D_\mu V^\nu \equiv \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho$

metric connection in torsion-free space $\quad \Gamma^\rho_{\mu\nu} \equiv \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\mu} + \partial_\nu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu}\right)$

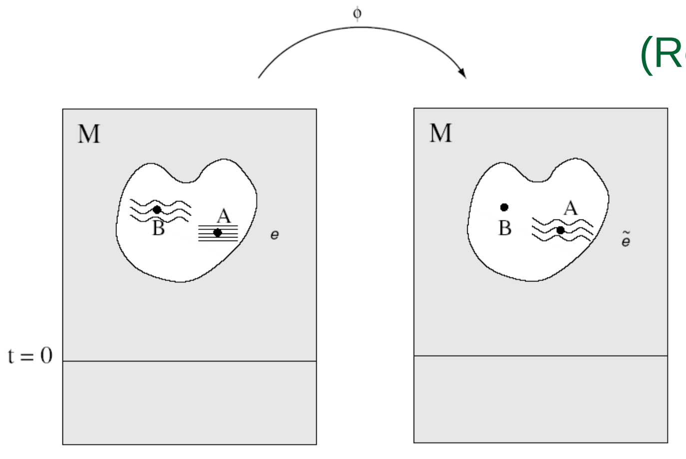
Riemann tensor ( ~ curvature) $\quad R^\lambda{}_{\mu\nu\rho} = \partial_\rho \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\sigma\rho}\Gamma^\sigma_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\rho} - \Gamma^\lambda_{\sigma\nu}\Gamma^\sigma_{\mu\rho}$

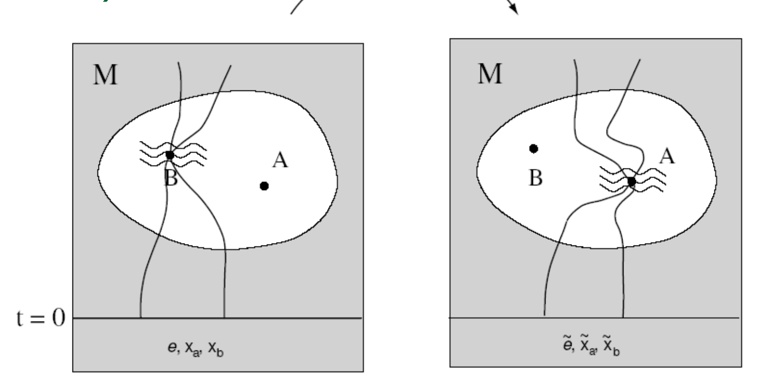


# I. Classical Gravity

- hole argument (1912 ~ 1915)

(Rovelli, 2004)




active diffeomorphism
A, B: spacetime point

passive diffeomorphism
A: intersecting point of two particle trajectories

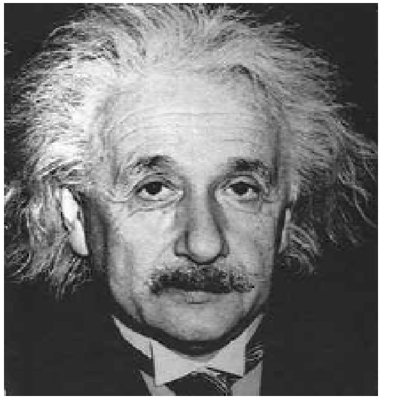
- There is no meaning in talking about the "physical" spacetime point in "vacuum".
- (Field equations + EOM of particles) must be generally covariant.

  If $e^I_\mu(x)$ is a solution, then $e'^I_\nu(y) = \dfrac{\partial x^\mu(y)}{\partial y^\nu} e^I_\mu(x(y))$ is also a solution.

# I. Classical Gravity

- Einstein equation (1915)

$$\boxed{G_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}}$$



geometry of spacetime    stress tensor of matter

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

(Einstein tensor)

"Matter tells space how to curve, and space tells matter how to move."  - J. A. Wheeler

"Einstein gravity"

$$S = -\frac{1}{16\pi G} \int d^4x \sqrt{-g}\, [R - 2\Lambda]$$

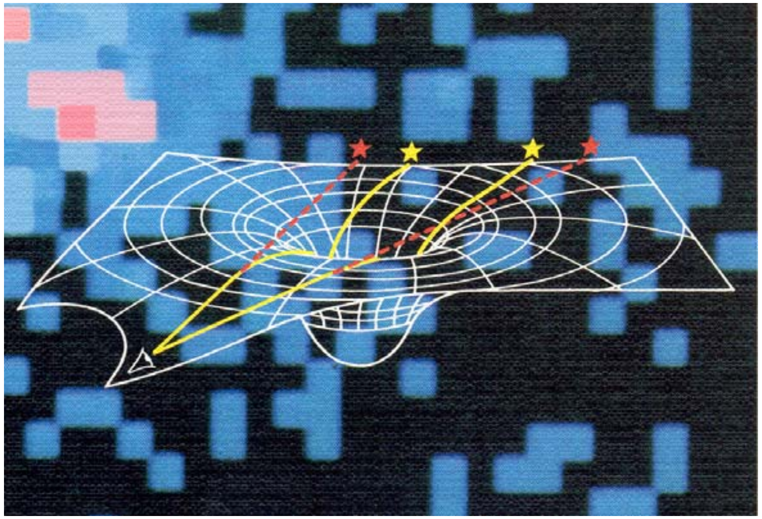
cosmological constant

"No metric, no nothing." - J. Stachel

# I. Classical Gravity

- Relativistic free particles go along timelike geodesics (back-reactions are neglected)

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0$$



form invariant under re-parameterization

$$\tau' = \alpha\,\tau$$

- Light goes in null geodesic $ds^2 = 0$ .

# I. Classical Gravity

- Gravitational field has no proper energy-momentum density, only psudo-tensors can be obtained (reference frame dependent).

- non-localizability: consequence of Einstein's equivalence principle - gravity cannot be detected at a point.

- Gravitational energy-momentum
  - and hence the energy-momentum of gravitating systems
  - and hence the energy momentum of all physical systems
  - is fundamentally non-local.

- Nester (2004): The integral of the boundary term in covariant Hamiltonian formalism gives quasi-local energy-momentum/flux associated with physically distinct boundary conditions.

# I. Classical Gravity

- Black holes

  E.g. Static, Spherically Symmetric vacuum solution

  (Schwarzschild 1916)

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

light: $ds^2 = 0$ .

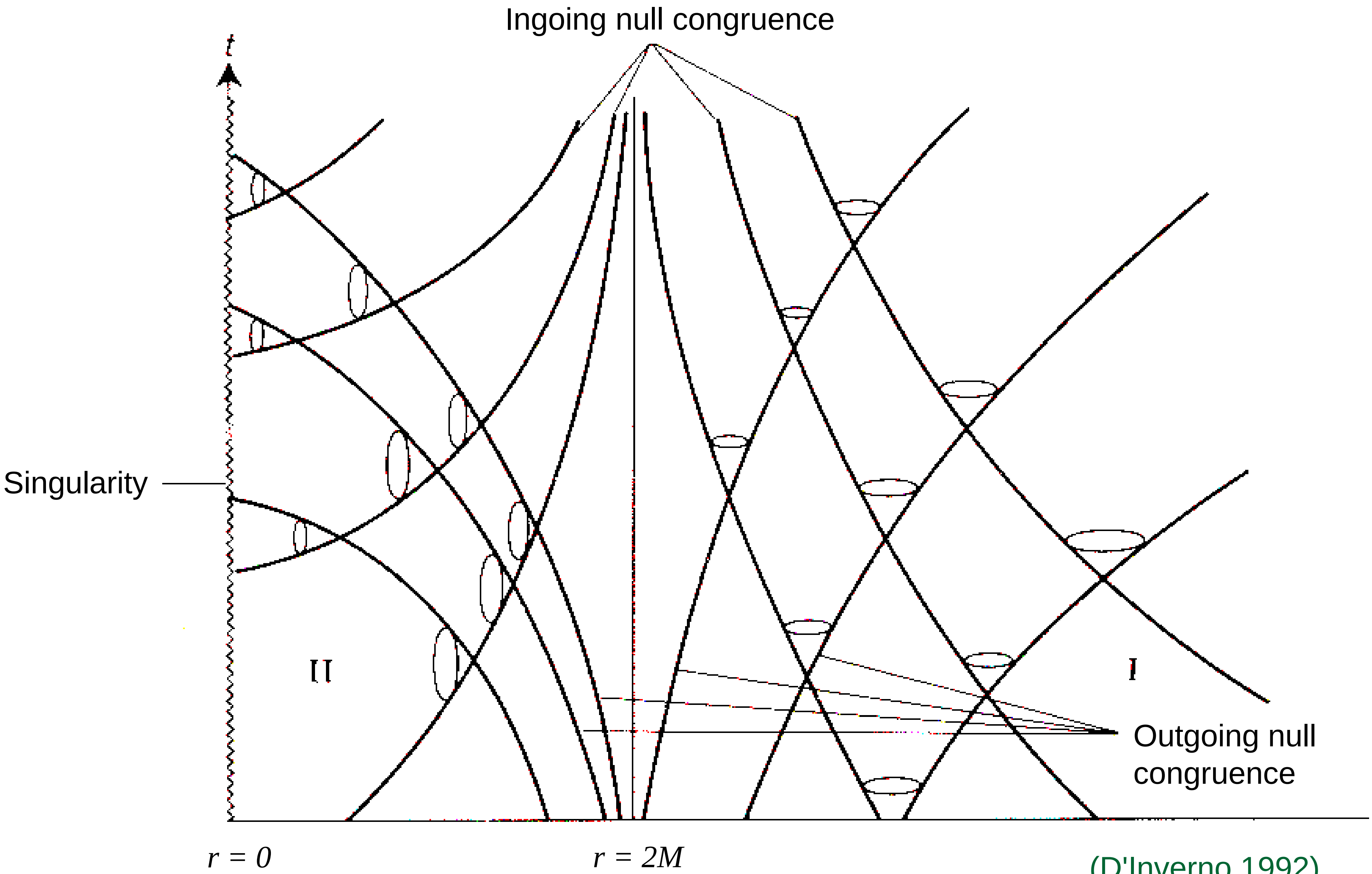
$$\Rightarrow \quad \frac{dr}{dt} = 1 - \frac{2M}{r} \qquad \rightarrow 0 \ \text{as}\ r \rightarrow 2M$$

Event horizon : coordinate singularity at $r=2M$

(pre-relativity: Laplace, $M$ : ADM mass of the black hole)

Classical Information behind the event horizon will never reach any observer sitting outside of the black hole.

"Black" hole: light signals got infinite red-shift when the source is approaching the event horizon.

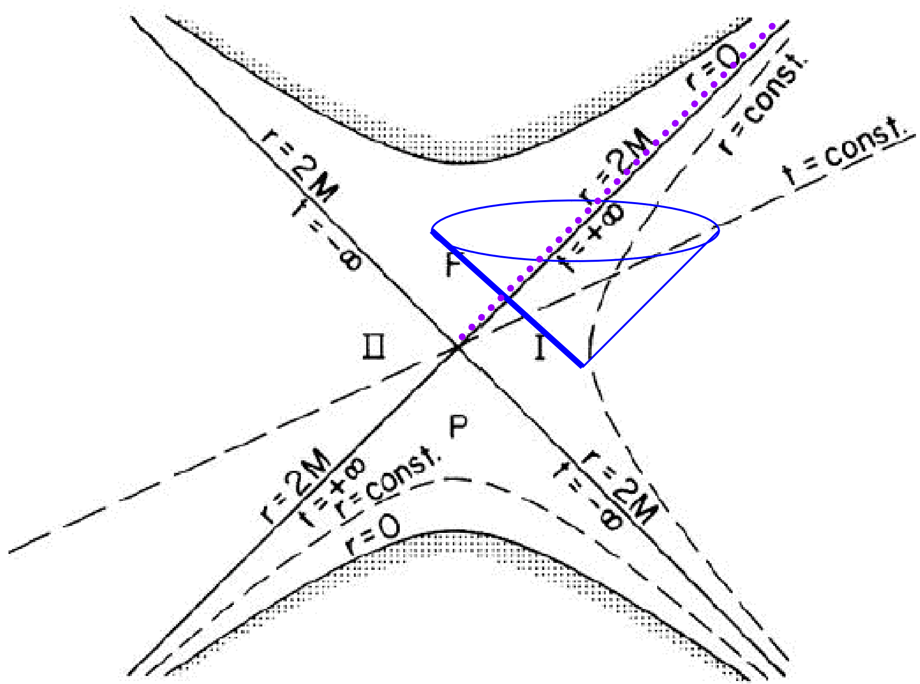


(D'Inverno 1992)

In Schwarzschild coordinate, only good for outside observers at rest.

# I. Classical Gravity

- Coordinate singularity at $r = 2M$ can be removed by a general coordinate transformation, e.g., Kruskal coordinate:



# I. Classical Gravity

- Relativistic Cosmology

1. Homogeneous and isotropic ansatz:

   Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right]$$

k = 1 (topology ~ R x $S^3$) closed, positive spatial curvature

k = 0 (~ $R^4$) open, zero spatial curvature

k = -1 (~ $R^4$) open, negative spatial curvature

2. Perfect fluid (Weyl) $T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}$ $\quad u^\mu = (1,0,0,0)$

3. Einstein equation $G_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}$

# I. Classical Gravity

- Singularity Theorem (Penrose, Hawking, Geroch, 1960s)
  - Gravitation is always attractive (~ energy conditions)

  If there is a trapped null surface and the energy density is nonnegative, then there exist geodesic of finite length which can't be extended.

- A collapsing star must evolve into a spacetime singularity (where the curvature diverges) in a black hole.
- The universe must start with a spacetime singularity (bounced universe is ruled out. )

Just like classical electrodynamics, GR predicts its own breakdown !!

# I. Classical Gravity

## Black hole thermodynamics (Bardeen, Carter, Hawking 1973)

- The Zeroth Law (equilibrium and temperature):

  Surface gravity $\kappa$ is constant over the event horizon.

- The First Law (change of internal energy):

  Any two neighboring stationary axisymmetric solutions containing a perfect fluid with circular flow and a central BH in it are related by

$$\delta M = \boxed{\frac{\kappa}{8\pi}\delta A} + \Omega_H \delta J_H + \int \Omega \delta dJ + \int \bar{\mu} \delta dN + \int \bar{\theta} \delta dS$$

- The Second Law (Entropy) $\sim T\,dS$

  Area of BH does not decrease with time,

$$\boxed{\delta A \geq 0}$$

- The Third Law (zero temperature)

  It is impossible by any procedure to reduce $\kappa$ to zero by a finite sequence of operations.

# I. Classical Gravity

- Gravitational wave

- indirect evidence: decrease of orbital period of binary pulsar PSR 1913+16

- attempts to detect directly:
  Weber's bar, LIGO, LISA,...

# II. Semiclassical Gravity

# II. Semiclassical Gravity

- Quantum fields in curved space

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G \langle T_{\mu\nu} \rangle$$

Geometry is classical and fixed, matter fields are quantized.

- Particle concept of quantum fields is obscure,
- Regularization and Renormalization of stress-tensor are tricky (choice of zero-point, anomaly)

- Semiclassical Gravity

  Geometry and expectation value of stress-tensor are solved consistently. Backreactions of the field are considered.

# II. Semiclassical Gravity

- Vacuum: ground state of the field

$$i\hbar \frac{\partial}{\partial t}\psi = \hat{H}\psi \qquad \hat{H} \sim \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} \left[\hat{n}_{\mathbf{k}} + \frac{1}{2}\right] \qquad \hat{n}_{\mathbf{k}} = \hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}$$

Vacuum state $|\,0\,\rangle$ is defined by $\hat{a}_{\mathbf{k}}|\,0\,\rangle = 0$, which implies $\langle\,0\,|\hat{n}_{\mathbf{k}}|\,0\,\rangle = 0$

Minkowski vacuum = no particle (field quanta) state for Minkowski observer.

(not always true in curved space)

- If there exist two natural directions of time,
one has two different Hamiltonians associated with them,
which define two vacuum states that may disagree with each other.

e.g. Rindler vacuum in Minkowski space vs. Minkowski vacuum

DeWitt: vacuum ~ new aether

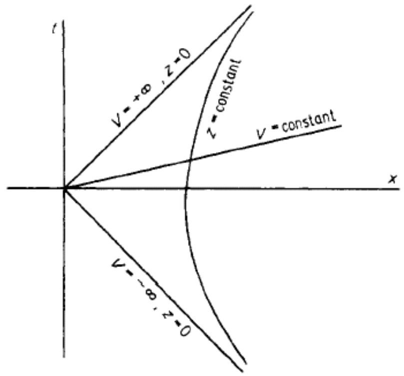
# II. Semiclassical Gravity

- In curved space, particle concept does not generally have universal significance.

  e.g. QFT in Rindler space (Fulling 1972, 1973; Davies 1975)

the other theory, and so on. The notion of a *particle* is completely different in the two theories. *The particles or quanta of the Rindler-Fock representation cannot be identified with the physical particles described by the usual quantum theory of the free field.*

The minimal conclusion which must be drawn from this observation is the following: *In the context of the general static universe treated in Sec. IIA, the particle concept does not have the full physical significance which it has in Minkowski space.* The theory of quantization in a static met-




- S.A. Fulling, PRD7, 2850 (1973)

Resolution:

- expectation values of tensorial, local-defined quantities

  (e.g. $<T_{\mu\nu}(x)>$) — depend on the choice of zero-point

- operational definitions — essentially observer-dependent

# II. Semiclassical Gravity

- Particle Creations in curved spacetime

Type1. Cosmological particle creation:

Time-varying background spacetime

$$\frac{d^2}{d\eta^2}\chi_{\mathbf{k}}(\eta) + \boxed{\omega_{\mathbf{k}}^2(\eta)}\chi_{\mathbf{k}}(\eta) = 0$$

- adiabatic vacuum

Type2. Black hole radiation, Unruh effect:

Out-modes are Infinitely red-shifted by static background

- trans-Planckian problem

# II. Semiclassical Gravity

Bekenstein (1973) :

BH entropy $S_{BH} = \text{const.} \frac{A}{l_P^2}$

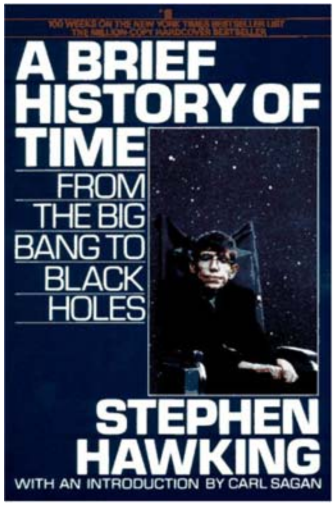
$l_P \equiv \hbar G / c^3$ Planck length

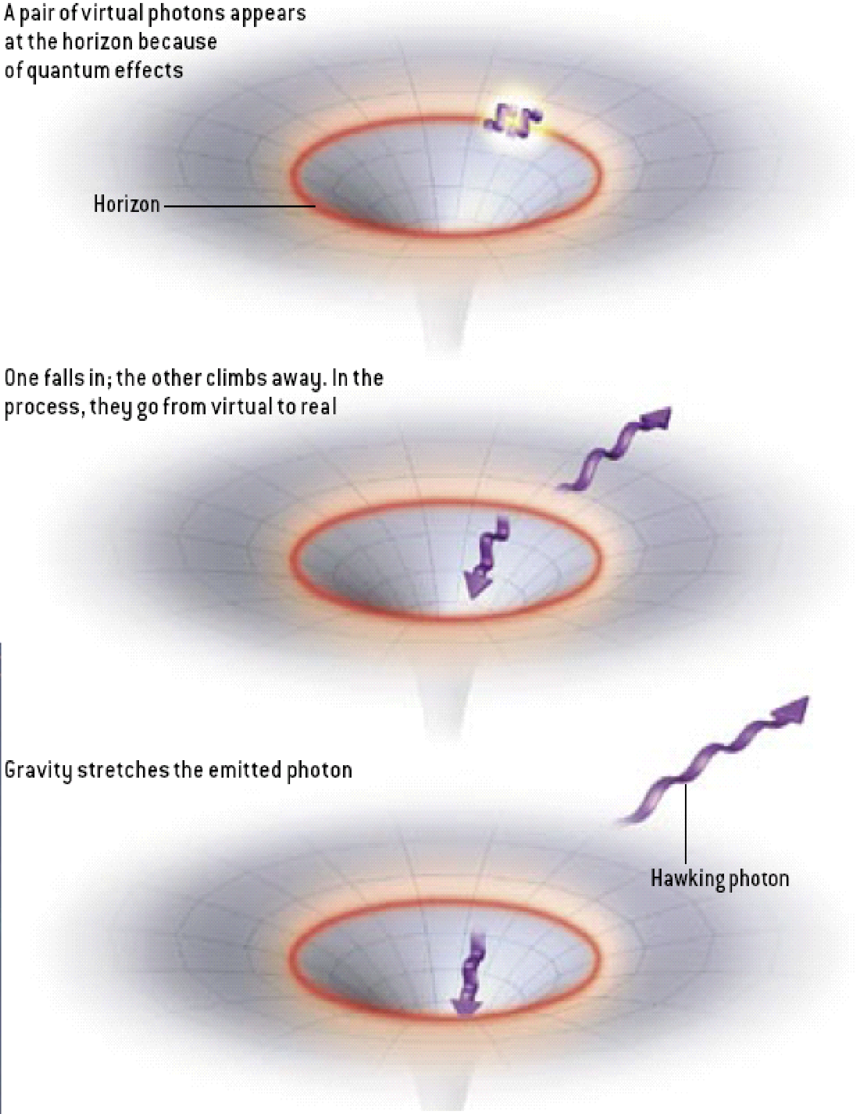
Black hole radiation (Hawking 1975)

w/temperature $T_{BH} = \frac{\hbar\kappa}{2\pi}$

thus $S_{BH} = \frac{A}{4 l_P^2}$

**Warning:**
Not confirmed by any experiment or observation yet !!



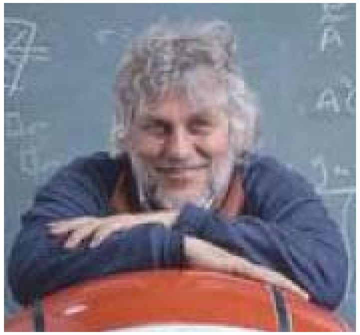




COPYRIGHT 2005 SCIENTIFIC AMERICAN, INC.

# II. Semiclassical Gravity

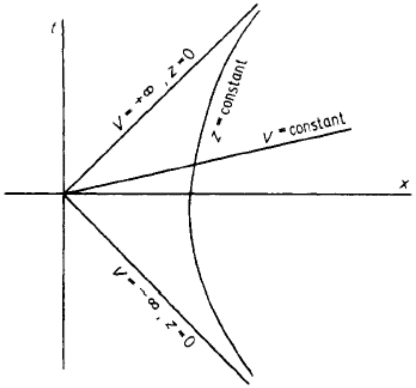
- Unruh effect (Unruh 1976)

  A **detector** uniformly accelerated in Minkowski vacuum will experience a thermal bath at **Unruh temperature**:



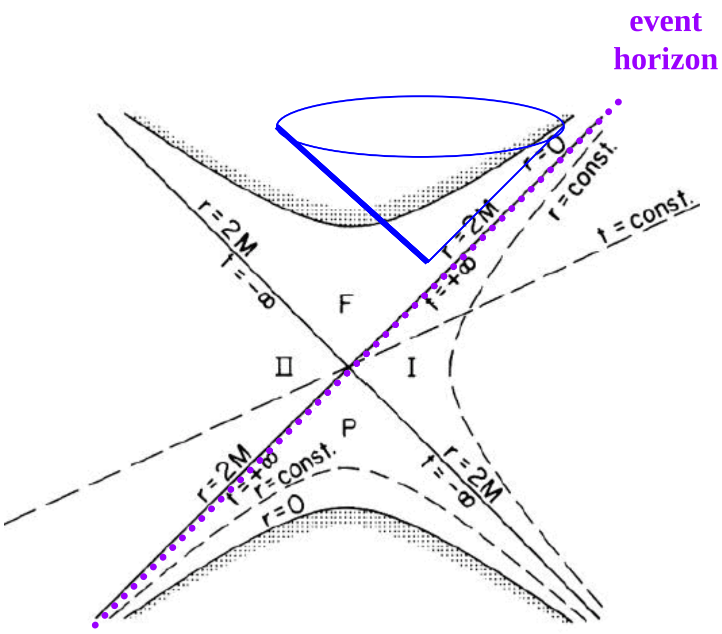
$$\boxed{T_U = \frac{\hbar a}{2\pi k_B c}}$$

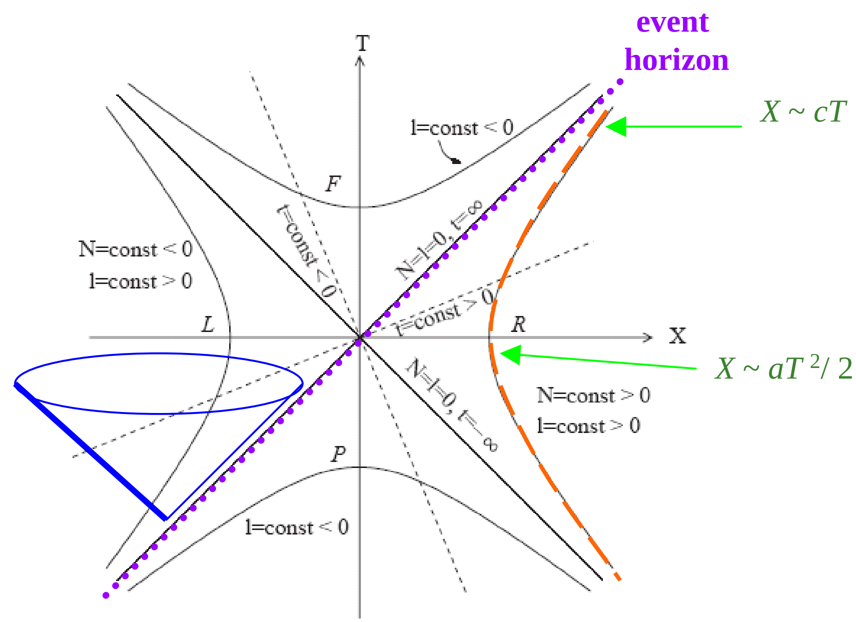
$T_U = 1\mathrm{K}$ for $a = 2.4 \times 10^{20}\ \mathrm{m/s^2}$




- **detector :** point-like object with internal degree of freedom coupled to a field
- uniform acceleration : $a_A a^A = a^2 = \text{constant}$ ($a$: proper acceleration)
- Minkowski vacuum: NO particle (field quanta) state of the field for Minkowski observer
- Implicitly assumed that $\hbar$ is invariant under coordinate transformations.

# I. Introduction: Black Hole Radiation and Unruh Effect




Schwarzchild BH in Kruskal coordinate
(Boulware 1975)

Minkowski space in Rindler coordinate
(Padamanabhan 2005)

Detector
fixed outside
a Black Hole

~
Equivalence
Principle

Detector
uniformly accelerated
in Minkowski space

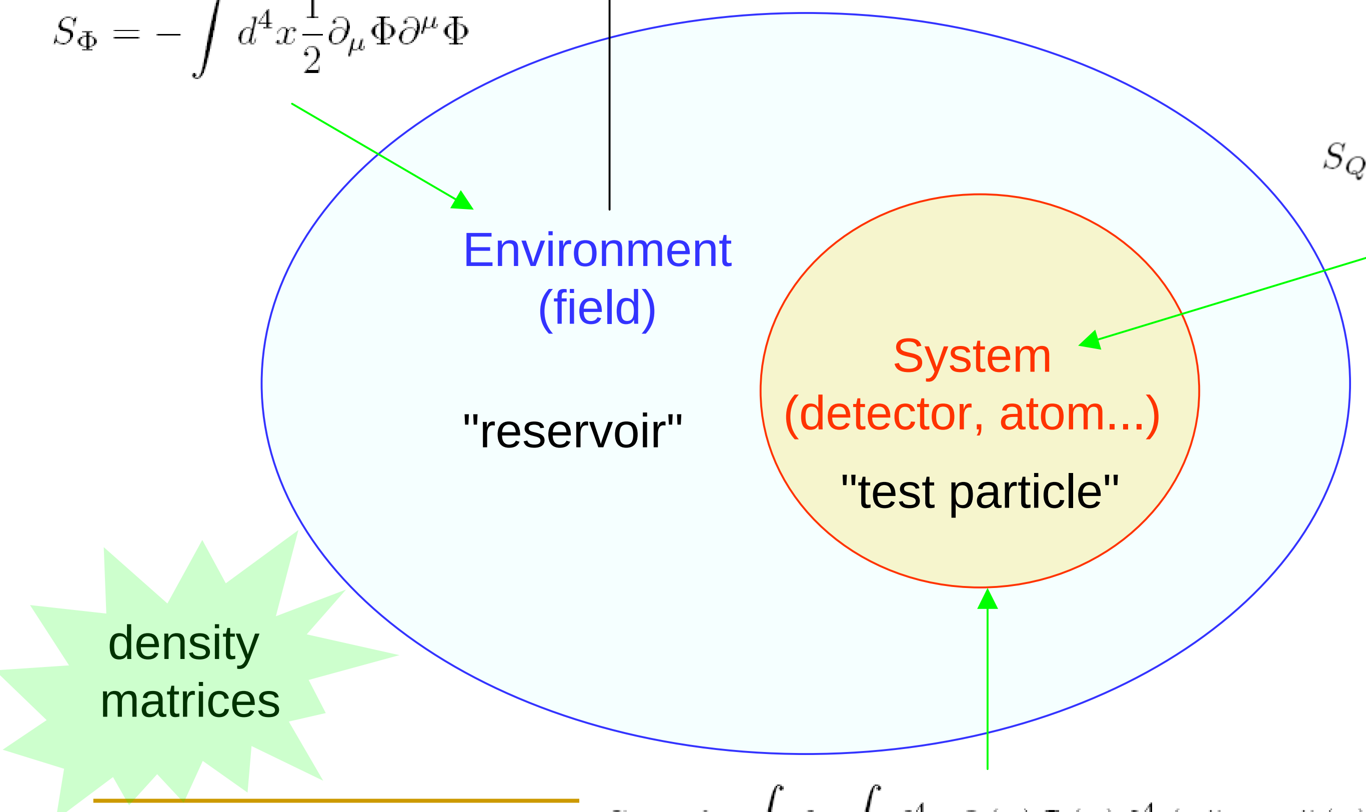
# II. Semiclassical Gravity

**Unruh effect**

A **detector** uniformly accelerated in Minkowski vacuum will experience a thermal bath at Unruh temperature.

$$S_\Phi = -\int d^4x \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi$$

$$S_Q = \int d\tau \frac{m_0}{2}\left[(\partial_\tau Q)^2 - \Omega_0^2 Q^2\right]$$




System can affect environment (back reaction).

$$S_I = \lambda_0 \int d\tau \int d^4x Q(\tau)\Phi(x)\delta^4\left(x^\mu - z^\mu(\tau)\right)$$

# II. Semiclassical Gravity

Unruh effect

A **detector** uniformly accelerated in Minkowski vacuum will experience a thermal bath at Unruh temperature.

The above "standard" statement is accurate
only at the initial moment of switching-on the coupling.
After that moment the statements make no sense beyond the ultra-weak coupling and ultra-high acceleration limits,
namely, beyond the Markovian (memory-less) regime.
(Cf. Photon-atom bound state in quantum optics)

Further, there exists non-trivial behavior even when $a = 0$.
(Lin and Hu 2007)

# III. Stochastic to Quantum Gravity

# III. Stochastic to Quantum Gravity

- Top-down approach:

  - String theory (lecture next week)

  - Canonical (Loop) quantum gravity

  - doubly special relativity ??

  Principles of quantum physics are good all the way down to (through) the Planck scale.

- Bottom-up approach

  - Stochastic gravity

  emergent relativity

Gravitation field may not be a fundamental field.

# III. Stochastic to Quantum Gravity

obtained by integrating out the matter part of the total density matrix

- Stochastic gravity

  Semiclassical Einstein-Langevin equation

$$\frac{1}{8\pi G}\left(G_{\mu\nu}[g+h]+\Lambda(g_{\mu\nu}+h_{\mu\nu})\right) = \langle \hat{T}^{R}_{\mu\nu}\rangle[g+h] + 2\xi_{\mu\nu}$$

where $g_{\mu\nu}$ is a solution of semiclassical Einstein equation,

$$G_{ab}[g]+\Lambda g_{ab} = 8\pi G\langle \hat{T}^{\mathrm{R}}_{ab}[g]\rangle$$

and $\xi_{\mu\nu}$ is the stochastic tensor field defined by correlators

$$\langle \xi_{ab}[g;x)\rangle_{\mathrm{s}} = 0, \qquad \langle \xi_{ab}[g;x)\xi_{cd}[g;y)\rangle_{\mathrm{s}} = N_{abcd}[g;x,y)$$

statistical average

$$N_{abcd}[g;x,y) = \frac{1}{2}\langle\{\hat{t}_{ab}[g;x),\hat{t}_{cd}[g;y)\}\rangle$$

$$\hat{t}_{ab}[g;x) \equiv \hat{T}_{ab}[g;x) - \langle \hat{T}_{ab}[g;x)\rangle$$

# III. Stochastic to Quantum Gravity

- Remarks:

  1. conservation: $\nabla^a \xi_{ab}[g;x) = 0$ ,  traceless: $g^{ab}\xi_{ab}[g;x) = 0$

  2. Einstein-Langevin equation is invariant under the gauge transf.

  $$h'_{ab} = h_{ab} + \nabla_a \zeta_b + \nabla_b \zeta_a$$

  where $\zeta_\mu$ is a stochastic vector field on the manifold.

- Applications
  - cosmological perturbations (structure formation)
  - backreaction to BH (event horizon fluctuations)

# III. Stochastic to Quantum Gravity

- Early efforts on full quantum gravity

  1. Quantum dynamics of metric (DeWitt, 1960s)
  2. metric perturbation as a spin-2, self-interacting gauge field on a background spacetime (Veltman and 't Hooft..., 1970s)

  - GR is not renormalizable at two-loops (Goroff & Sagnotti, 1985)

    we need the counterterm

$$\Gamma^{(2)}_{\rm div} = \frac{1}{\epsilon}\frac{209}{2880}\frac{1}{(16\pi^2)^2}\int d^4x\,\sqrt{g}\,C_{\mu\nu\rho\sigma}C^{\rho\sigma\lambda\tau}C_{\lambda\tau}{}^{\mu\nu}$$

  - higher-derivatives($R^2$ gravity): 2-loop finite,

    Supergravity : 2-loops finite

# III. Stochastic to Quantum Gravity

- Geometrodynamics: Recall Classical Canonical Gravity

*Canonical analysis in ADM variable*

✦ Einstein-Hilbert action [in metric variables]

$$I[g_{\mu\nu}] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(\mathcal{R} - 2\Lambda)$$

✦ ADM Decomposition: introduce a foliation of spacetime $\mathcal{M} = \Sigma \times \mathrm{R}$

- $g_{\mu\nu} \to q_{ab}$, $N_a$ : shift function, $N$: laspe function.
- $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2(dx^0)^2 + q_{ab}(dx^a + N^a dx^0)(dx^b + N^b dx^0)$

$$g_{\mu\nu} = \begin{pmatrix} q_{ab}N^aN^b - N^2 & q_{ab}N^a \\ q_{ab}N^b & q_{ab} \end{pmatrix}, g^{\mu\nu} = \begin{pmatrix} -1/N^2 & N^a/N^2 \\ N^b/N^2 & q^{ab} - N^aN^b/N^2 \end{pmatrix}$$

✦ After performing the Legendra transformation:

$$I[q_{ab}, \pi^{ab}, N_a, N] = \frac{1}{16\pi}\int dt \int_\Sigma d^3x[\pi^{ab}\dot{q}_{ab} - \mathcal{H}]$$

- $\pi^{ab} = -\frac{\sqrt{q}}{16\pi G}(K^{ab} - Kq^{ab})$ : momenta canonically conjugate to $q_{ab}$, $K_{ab} = \frac{1}{2N}(-\partial_0 q_{ab} + \nabla_a N_b + \nabla_b N_a)$ : extrinsic curvature.



Special thanks to Dr. C. H. Chou

# III. Stochastic to Quantum Gravity

$$S[q_{ab}, \pi^{ab}, N_a, N] = \frac{1}{16\pi} \int dt \int_\Sigma d^3x \, [\pi^{ab} \dot{q}_{ab} - \mathcal{H}]$$

$$\mathcal{H}(q_{ab}, \pi^{ab}, N_a, N) = N^a H_a(q_{ab}, \pi^{ab}) + N H(q_{ab}, \pi^{ab})$$

- Super-momentum constraint: $H_a(q_{ab}, \pi^{ab}) = -\frac{2}{16\pi G} \nabla_b \pi^b{}_a \quad (= 0)$
- Super-Hamiltonian constraint:

$$\begin{aligned} H(q_{ab}, \pi^{ab}) &= \frac{8\pi G}{\sqrt{q}} (q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd}) \pi^{ab}\pi^{cd} - \frac{\sqrt{q}}{16\pi G}(R(q) - 2\Lambda) \\ &= \frac{\sqrt{q}}{16\pi G}[K^{ab}K_{ab} - K^2 - R(q) + 2\Lambda] \quad (= 0) \end{aligned}$$

✦ Degrees of freedom of GR in 4D:

6 pairs $(q_{ab}, \pi^{ab})$ subject to 4 constraints = 2 FIELD d.o.f.

✦ The Poisson brackets are

$$\begin{aligned} \{\pi^{ab}(x), q_{cd}(y)\} &= 16\pi \delta^a_{(c}\delta^b_{d)} \delta(x, y), \\ \{q_{ab}(x), q_{cd}(y)\} &= \{\pi^{ab}(x), \pi^{cd}(y)\} = 0 \end{aligned}$$

✦ Phase space variables: $(q_{ab}, \pi^{cd})$

# III. Stochastic to Quantum Gravity

## *Canonical Quantization of GR*

- Does not require background spacetime (background independence)
- Can be used for strong and weak GR fields.
- Conjugate variables:

$$\{q_{ab}(\vec{x}), \pi^{cd}(\vec{y})\}_{P.B.} = \frac{1}{2}(\delta_a{}^c\delta_b{}^d + \delta_b{}^c\delta_a{}^d)\delta^3(\vec{x}-\vec{y})$$

- Canonical Quantization :

$$\{\ ,\ \}_{P.B.} \to \frac{1}{i\hbar}[\ ,\ ]; \quad q_{ab} \to \hat{q}_{ab}, \quad \pi^{ab} \to \hat{\pi}^{ab}$$

- Metric representation: Wavefunction $\Psi[q_{ab}]$
  - $\hat{q}_{ab}\Psi[q_{ab}] = q_{ab}\Psi[q_{ab}]$ ; $\quad \hat{\pi}^{ab}\Psi[q_{ab}] = \frac{1}{i\hbar}\frac{\delta}{\delta q_{ab}}\Psi[q_{ab}]$
- Constraints (First Class) (Dirac Quantization):

$$\hat{H}_a(\hat{q}_{ab}, \hat{\pi}^{ab})\Psi[q_{ab}] = \hat{H}_a(\hat{q}_{ab}, \tfrac{1}{i\hbar}\tfrac{\delta}{\delta q_{ab}})\Psi[q_{ab}] = 0$$

$\Leftrightarrow$ $\Psi[q'_{ab}] = \Psi[q_{ab}]$ if $q_{ab}$ is related to $q'_{ab}$ by a 3-dimensional diffeomorphism

$\Leftrightarrow$ $\Psi[\mathcal{G}]$. 3-geometry $\mathcal{G} \in$ SUPERSPACE:

Space of all 3-geometries (equivalence class of 3-metrics) $q'_{ab} \sim q_{ab}$

iff they are related by 3-dim. general coordinate transformation.

# III. Stochastic to Quantum Gravity

✦ Quantum super-Hamiltonian Constraint: Wheeler-DeWitt Equation

$$\hat{H}\Psi[q_{ab}] \sim \text{"}[G_{abcd}\frac{\delta}{\delta q_{ab}}\frac{\delta}{\delta q_{cd}} + \sqrt{q}(R(q) - 2\Lambda)]\text{"}\,\Psi[\mathcal{G}] = 0$$

Supermetric $G_{abcd} = \frac{8\pi G}{\sqrt{q}}(q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd})$.

Symbolically,

$$[\frac{\delta^2}{\delta\mathcal{G}^2} + (R(q) - 2\Lambda)]\Psi[\mathcal{G}] = 0$$

✦ Technical issues:

Ordering, Regularization, Anomalies, Explicit Solutions, of Wheeler-DeWitt Equation.

# III. Stochastic to Quantum Gravity

✦ Important conceptual issues: Where/what is physical "time" in Quantum Gravity?

- Note: $x^0$ is not "time". Theory is reparametrization invariant. $H$ does not generate "time" translation: $\exp\left(\frac{-ix^0H}{\hbar}\right)\Psi[\mathcal{G}] = \Psi[\mathcal{G}]$.

✦ B. S. DeWitt [Phys. Rev. **160**, 1113 (1967)]:

Supermetric $G^{abcd}\delta q_{ab}\delta q_{cd} = -(\delta\xi)^2 + (\frac{3}{32})\xi^2\bar{G}_{AB}\delta\xi^A\delta\xi^B$ i.e.

$G^{\{ab\}\{cd\}} = diag(-1, \frac{3}{32}\xi^2\bar{G}_{AB})$ ; $A, B = 1, 2, 3, 4, 5.$

$\bar{G}_{AB}$: positive-definite $\Rightarrow$ supermetric has signature $(-,+,+,+,+,+)$.

" $-$ " direction is associated with "intrinsic time" $\xi = \sqrt{32/3}(\det q)^{1/4}$.

Superspace is hyperbolic.

Super-Hamiltonian constraint has "dynamical" content.

Wheeler-DeWitt Equation:

$$"[-\frac{\delta^2}{\delta\xi^2} + \frac{32}{3\xi^2}\bar{G}^{AB}\frac{\delta}{\delta\xi^A}\frac{\delta}{\delta\xi^B} + \frac{3\xi^2}{32}(R(q) - 2\Lambda)]"\,\Psi[\mathcal{G}] = 0$$

In simple homogeneous isotropic cosmological models (e.g. of minisuperspace), $\xi \propto [a(t)]^{3/2}$ ($a =$ expansion scale factor).

# III. Stochastic to Quantum Gravity

- Full Quantum Gravity: New variables

*The triad formulation*

✦ To use a triad (a set of 3 1-forms at each point in $\Sigma$)

$$q_{ab} = e_a^i e_b^j \delta_{ij}$$

- Densitized triad: $E_i^a = \frac{1}{2}\epsilon^{abc}\epsilon_{ijk}e_b^j e_c^k$
- Additional 3 (Gauss) constraints: $G_i(E_j^a, K_a^j) = \epsilon_{ijk}E^{aj}K_a^k = 0$

✦ With new variables, the action of GR becomes

$$I[E_j^a, K_a^j, N_a, N, N^j] = \frac{1}{8\pi}\int dt \int_\Sigma d^3x [E_i^a \dot{K}_a^i - N^b H_b(E_j^a, K_a^j) - N H(E_j^a, K_a^j) - N^i G_i(E_j^a, K_a^j)]$$

The sympletic structure now becomes

$$\begin{aligned} \{E_j^a(x), K_b^i(y)\} &= 8\pi\delta_b^a\delta_j^i\delta(x,y), \\ \{E_j^a(x), E_i^b(y)\} &= \{K_a^j(x), K_b^i(y)\} = 0 \end{aligned}$$

# III. Stochastic to Quantum Gravity

## *The Ashtekar-Barbero connection variables*

✦ There is a natural $so(3)$-connection (spin-connection $\Gamma^i_a$) that defines the notion of covariant derivative compatible with the dreibein

$$\partial_{[a} e^i_{b]} + \epsilon^i{}_{jk} \Gamma^j_{[a} e^k_{b]} = 0$$

- Ashtekar-Barbero variable: $A^i_a = \Gamma^i_a + \gamma K^i_a$
- $\gamma$ : Immirzi parameter, $\gamma \in \mathrm{R} - \{0\}$.

✦ With the connection variables, the action becomes

$$I[E^a_j, A^j_a, N_a, N, N^j] = \frac{1}{8\pi} \int dt \int_\Sigma d^3x [E^a_i \dot{A}^i_a - N^b H_b(E^a_j, A^j_a) - N H(E^a_j, A^j_a) - N^i G_i(E^a_j, A^j_a)]$$

- $H_b(E^a_j, A^j_a) = E^a_j F^j_{ab} - (1+\gamma^2) K^i_b G_i = 0$
- $H(E^a_j, A^j_a) = \frac{E^a_i E^b_j}{\sqrt{\det(E)}} (\epsilon^{ij}{}_k F^k_{ab} - 2(1+\gamma^2) K^i_{[a} K^j_{b]}) = 0$
- $G_i(E^a_j, A^j_a) = D_a E^a_i = 0$

# III. Stochastic to Quantum Gravity

- where $F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + \epsilon^i{}_{jk} A^j_a A^k_b$ and $D_a E^a_i = \partial_a E^a_i + \epsilon_{ij}{}^k A^j_a E^a_k$

✦ The Poisson bracket of the new variables are

$$\begin{aligned} \{E^a_j(x), A^i_b(y)\} &= 8\pi\gamma\delta^a_b\delta^i_j\delta(x,y), \\ \{E^a_j(x), E^b_i(y)\} &= \{A^j_a(x), A^i_b(y)\} = 0 \end{aligned}$$

✦ Phase space variables: $(A^i_a, E^b_j)$

✦ Series of (Canonical) transformations:

Metric variables: $(q_{ab}, \pi^{ab})$

$\rightarrow (e_{ai}, \pi^{ai})+$ 3 gauge constraints (Gauss' Law)

$\rightarrow (E^a_i, K^i_a)+$ Gauss' Law

$\rightarrow (E^a_i, A^{i\,-}_a = \Gamma^i_a - iK^i_a)+$ Gauss' Law (Ashtekar Variable)

$\rightarrow (E^a_i, A^i_a = \Gamma^i_a + \gamma K^i_a)+$ Gauss' Law (Ashtekar-Barbero Variable)

(related discussion: C.H.C, R.H. Tung, H. L. Yu, PRD **72**, 064016 (2005))

# III. Stochastic to Quantum Gravity

*Conceptual Breakthroughs*

- Distinction between geometrodynamics and gauge dynamics is bridged. Identify $E_j^a$ as the momentum conjugate to the gauge potential $A_a^i$;
  $\Rightarrow (E_j^a, A_a^i)$ phase space identical to Yang-Mills Theory.
- Quantum States can be wavefunctions in A-representation $\Psi[A]$, with $E_i^a = (\frac{8\pi G\hbar}{c^3})\frac{\delta}{\delta A_a^i}$. All manipulations done on gauge variables.

# III. Stochastic to Quantum Gravity

## *Technical Breakthroughs*

- Constraints much simpler:
- Exact solution found (e.g. Chern-Simons state, in field theory variables)
- Loop variables: Wilson loops: holonomy elements.
  - Gauss's constraint solved by $\Psi$[Wilson loops in $A$] ;
  - $H_a = 0$ solved by $\Psi$[knot classes of Wilson loops in $A$].
- Super-Hamiltonian constraint still difficult, but can be made well-defined:
  - Volume $V$ and area $\mathcal{A}$ operators : well-defined operators acting on loop and spin network states and have discrete spectra.
- Derivation of horizon entropy, both for black hole and cosmological horizons.
  - Black hole evaporation via transition from higher $\mathcal{A}$ states to lower $\mathcal{A}$ states.

# III. Stochastic to Quantum Gravity

*Quantization of area*

- Rovelli and Smolin (1994); Ashtekar, Lewandowski et al (1995): given a surface

$$A(\mathcal{S}) = \int_{\mathcal{S}} \sqrt{n_a E_i^a n_b E_i^b} d^2\sigma$$

- The quantum area spectrum is

$$A(\mathcal{S})|S\rangle = 8\pi\gamma \sum_P \sqrt{j_P(j_P+1)}|S\rangle$$

- Why is geometry discrete ?
  - The value of a triad in a given point is conjugate to the connection in the same point but Poisson commute with values of the connection in any other points.
  - The flux operator will only notice intersection points.
  - The eigenvalues of the flux operator: discrete.
  - Triad $\longrightarrow$ metric $\longrightarrow$ length, area, volume.... Geometry is discrete.
  - The result is topological and background independent.
  - The spin of the lines of a spin network can be viewed as "quanta of area".

# III. Stochastic to Quantum Gravity

- Remarks on Loop Quantum Gravity

  - loop (holonomy) variable:

    diffeomorphism (general coordinate transformation) invariant

    and background independent

  - finite

  - Lorentz symmetry is violated below Planck scale (aether)

- Comments

  - cannot predict anything at large length scale or in low-energy limit (Baez, 2004), or

    cannot show the ground state is semiclassical, or an emergence of classical spacetime (Smolin, 2005)

# IV. Concluding Remarks

# IV. Concluding Remarks

- Even if QG theory comes true, we still have outstanding work in theoretical physics:

    quantum to classical,

    micro to macro

- 3 pillars of fundamental physics:

    general relativity, quantum field theory, statistical mechanics

    ......BH entropy

- Revolution is still on-going.